

H^∞ -calculus for semigroup generators on BMO

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Abstract

We prove that the negative generator L of a semigroup of positive contractions on L^∞ has bounded $H^\infty(S_\eta)$ -calculus on $\text{BMO}(\sqrt{L})$ for any angle $\eta > \pi/2$, provided L satisfies Bakry-Émery's $\Gamma^2 \geq 0$ criterion. Our arguments only rely on the properties of the underlying semigroup and works well in the noncommutative setting. A key ingredient of our argument is a quasi monotone property for the subordinated semigroup $T_{t,\alpha} = e^{-tL^\alpha}$, $0 < \alpha < 1$, that is proved in the first part of this article.

Introduction

Let $\Delta = -\partial_x^2$ be the negative Laplacian operator on \mathbb{R}^n . The associated Poisson semigroup of operators $P_t = e^{-t\sqrt{\Delta}}$, $t \geq 0$ has many nice properties that make it a very useful tool in the classical analysis. In particular, the Poisson semigroup has a quasi monotone property that there exist constants $c_{r,j}$ such that, for any nonnegative function $f \in L^1(\mathbb{R}^n, \frac{1}{1+|x|^2} dx)$,

$$|t^j \partial_t^j P_t f| \leq c_{r,j} P_{rt} f, \quad (1)$$

for any $0 < r < 1, j = 0, 1, 2, \dots$. As a first result of this article, we show that the quasi monotone property (1) extends to all subordinated semigroups $T_{t,\alpha} = e^{-tL^\alpha}$ for all $0 < \alpha < 1$ if L generates a semigroup of positive preserving operators on a Banach lattice X . The case of $0 < \alpha \leq \frac{1}{2}$ is easy and is previously known because of a precise subordination formula (see e.g. [24, 20]).

Functional calculus is a theory of studying functions of operators. The so-called H^∞ -calculus is a generalization of the Riesz-Dunford analytic functional calculus and defines $\Phi(L)$ via a Cauchy-type integral for an (unbounded) sectorial operator L and a function Φ that is bounded and holomorphic in a sector S_η of the complex plane. L is said to have the bounded H^∞ -calculus property if the so-defined $\Phi(L)$ extends to bounded operators on X and $\|\Phi(L)\| \leq c\|\Phi\|_\infty$ for all such Φ 's. The theory of bounded H^∞ -calculus has developed rapidly in the last thirty years with many applications and interactions with harmonic

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analysis, Banach space theory, and the theory of evolution equations, starting with A. McIntosh's seminal work in 1986 ([23], [14],[22],[31]).

It is a major task in the study in the bounded H^∞ -calculus theory to determine which operators have such a strong property. Cowling, Duong, and Hieber & Prüss ([7, 11, 16]) prove that the infinitesimal generator of a semigroup of positive contractions on L^p , $1 < p < \infty$ always has the bounded $H^\infty(S_\eta)$ -calculus on L^p for any $\eta > \frac{\pi}{2}$. When the semigroup is symmetric, the angle can be reduced to $\eta > \omega_p = |\frac{\pi}{2} - \frac{\pi}{p}|$ by interpolation. It is not surprising that this result fails for L^∞ in general. One may want to seek a BMO-type space that could be an appropriate alternative for $p = \infty$ case. The main theorem of this article states that the negative generator L of a semigroup of positive contractions on L^∞ always has bounded H^∞ -calculus on the space $\text{BMO}(\sqrt{L})$, provided L satisfies Bakry-Émery's Γ^2 criterion.

The classical BMO norm of a function $f \in L^1(\mathbb{R}^n, \frac{1}{1+|x|^2}dx)$ can be defined as

$$\|f\|_{\text{BMO}(\sqrt{\Delta})} = \sup_{t>0} \left\| e^{-t\sqrt{\Delta}} \left| f - e^{-t\sqrt{\Delta}} f \right|^2 \right\|_{L^\infty}^{\frac{1}{2}}. \quad (2)$$

BMO spaces associated with semigroup generators have been intensively studied recently (e.g. [12] and the after-works). When a cubic-BMO is available, one can often compare it with the semigroup BMO and they are equivalent in many cases. In this article, we follow the approach from [20, 24] and consider the $\text{BMO}(\sqrt{L})$ -(semi)norm defined similarly to (2), merely replacing Δ with the studied semigroup generator L . The corresponding space $\text{BMO}(\sqrt{L})$ interpolates well with L^p -spaces when the semigroup is symmetric Markovian (see Lemma 11).

Under the assumptions of our main theorem, we also study semigroup-BMO spaces $\text{BMO}(L^\alpha)$, $0 < \alpha < 1$ and prove that they are all equivalent. We further prove that the imaginary power L^{is} is bounded on the associated semigroup-BMO space $\text{BMO}(L^\alpha)$ with a bound $\lesssim (1 + |s|)^{\frac{3}{2}} \exp(|\frac{\pi s}{2}|)$ (see (71),(72)). This complements Cowling's L^p -estimate (see [7, Corollary 1]) and fixes a mistake in [20] (see the Remark in Section 3).

The related topics and estimates on semigroup generators have been studied with geomtric/metric assumptions on the underlying measure space. This article is from a functional analysis point of view and tries to obtain a general result by abstract arguments. Cowling and Hieber/Prüss's method for their H^∞ -calculus results on L^p is based on the transference techniques of Coifman and Weiss, which does not work for non-UMD Banach spaces, such as BMO. Our method is to consider the fractional power of the generator to take the advantages of the quasi-monotone property (1). Our argument works well for the noncommutative case, that is, for L that generates a semigroup of completely positive contractions on a semifinite von Neumann algebra.

We analyze a few examples to illustrate our results at the end of the article. We use c for an absolute constant which may differs from line to line.

1 The complete monotonicity of a difference of exponential power functions

A nonnegative C^∞ -function $f(t)$ on $(0, \infty)$ is *completely monotone* if

$$(-1)^k \partial_t^k f(t) \geq 0$$

for all t . Easy examples are $f(t) = e^{-\lambda t}$ for any $\lambda > 0$. It is well-known that complete monotonicity is preserved by addition, multiplication, and taking pointwise limits. So the Laplace transform of a positive Borel measures on $[0, \infty)$, which is an average of $e^{-\lambda t}$ in λ , is completely monotone. The Hausdorff-Bernstein-Widder Theorem says that the reverse is also true; namely that a function is completely monotone if and only if it is the Laplace transform of a positive Borel measures on $[0, \infty)$. In particular, $g_s(t) = e^{-st^\alpha}$ is completely monotone and is the Laplace transform of a positive integrable C^∞ function $\phi_{s,\alpha}$ on $(0, \infty)$ for all $s > 0, 0 < \alpha < 1$.

$$e^{-st^\alpha} = \int_0^\infty e^{-\lambda t} \phi_{s,\alpha}(\lambda) d\lambda = \int_0^\infty e^{-s^{\frac{1}{\alpha}} \lambda t} \phi_{1,\alpha}(\lambda) d\lambda. \quad (3)$$

The function $\phi_{s,\alpha}$ is uniquely determined by the inverse Laplace transform

$$\phi_{s,\alpha}(\lambda) = s^{-\frac{1}{\alpha}} \phi_{1,\alpha}(s^{-\frac{1}{\alpha}} \lambda) = \mathcal{L}^{-1}(e^{-sz^\alpha})(\lambda) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{z\lambda} e^{-sz^\alpha} dz, \quad (4)$$

for $\sigma > 0, \lambda > 0$. The derivative $\partial_s \phi_{s,\alpha}$ is again an integrable function (see e.g. [32, page 263]), and

$$-t^\alpha e^{-st^\alpha} = \int_0^\infty e^{-\lambda t} \partial_s \phi_{s,\alpha}(\lambda) d\lambda. \quad (5)$$

The properties of $\phi_{s,\alpha}$ are important in the study of the fractional powers of semigroup generators.

The goal of this section is to prove a few pointwise inequalities for $\phi_{s,\alpha}$, which will be used in the next section. For that purpose, we first prove the complete monotonicity of several variants of e^{-st^α} .

For $k, n \in \mathbb{N}, 1 \leq k \leq n$, let $a_k^{(n)}$ be the real coefficients in the expansion

$$\frac{d^n}{dt^n} e^{-t^\alpha} = (-1)^n \sum_{k=1}^n a_k^{(n)} t^{-n+k\alpha} e^{-t^\alpha}.$$

It is easy to see that

$$\frac{d^n}{dt^n} e^{-ct^\alpha} = (-1)^n \sum_{k=1}^n c^k a_k^{(n)} t^{-n+k\alpha} e^{-ct^\alpha}.$$

We define $a_k^{(n)} = 0$ if $k > n$ or $k \leq 0$. The proof of the following lemma is simple and elementary. We leave it for the reader to verify.

Lemma 1. The $a_k^{(n)}$'s satisfy the relation

$$a_k^{(n+1)} = (n - k\alpha)a_k^{(n)} + \alpha a_{k-1}^{(n)} \quad (6)$$

for all $k \in \mathbb{Z}, n \in \mathbb{N}$.

Lemma 2. Let $K_i, i = 1, 2$ be the first integer such that $\frac{K}{K+i} \geq \alpha$. Then, for all $j \in \mathbb{Z}, n \in \mathbb{N}$, we have

$$\begin{aligned} a_{k+j}^{(n)} - (j+1)a_{k+j+1}^{(n)} &\geq 0 \quad \text{if } k \geq K_1 \\ (j+1)(a_{k+j+1}^{(n)} - (j+2)a_{k+j+2}^{(n)}) &\leq a_{k+j}^{(n)} - (j+1)a_{k+j+1}^{(n)} \quad \text{if } k \geq K_2 \end{aligned} \quad (7)$$

Proof. We only need to prove the case $j \geq 0$. Let D be the right derivative for discrete functions: $Df = f(j+1) - f(j)$. It is easy to see that the product rule holds $D(jf)(j) = jD_j f(j) + f(j+1)$. Fix $k \in \mathbb{Z}$. Let

$$f_n(j) = a_{k+j}^{(n)} j! \quad (9)$$

for $j \geq 0$, where we use the convention that $0! = 1$. By (6), we have

$$f_{n+1}(j) = (n - (k+j)\alpha)f_n(j) + \alpha j f_n(j-1),$$

for all $j \geq 1$ and $f_{n+1}(0) = (n - k\alpha)f_n(0) + \alpha a_{k-1}^{(n)}$. Taking the discrete derivative on both sides, we get

$$\begin{aligned} Df_{n+1}(j) &= (n - (k+j)\alpha)Df_n(j) - \alpha f_n(j+1) + \alpha j Df_n(j-1) + \alpha f_n(j) \\ &= (n - (k+j+1)\alpha)Df_n(j) + \alpha j Df_n(j-1). \end{aligned} \quad (10)$$

for $j \geq 1$ and $Df_{n+1}(0) = (n - (k+1)\alpha)Df_n(0) - \alpha a_{k-1}^{(n)}$. By induction, we get

$$D^i f_{n+1}(j) = (n - (k+j+i)\alpha)D^i f_n(j) + \alpha j D^i f_n(j-1). \quad (11)$$

for all $i \geq 1, j \geq 1$ and $D^i f_{n+1}(0) = (n - (k+i)\alpha)D^i f_n(0) + (-1)^i \alpha a_{k-1}^{(n)}$.

Let $k = K_1$ in (9). Note that the condition $Df_n(j) \leq 0$ trivially holds for $n \leq K_1 + j$ because $a_i^{(j)} = 0$ for $i > j$. In particular, $Df_n(j) \leq 0$ for all $j \geq 0, n = K_1$. We apply induction on n . Assume $Df_n(j) \leq 0$ holds for all $j \geq 0$. The equality (10) implies that $Df_{n+1}(j) \leq 0$ for all $j \geq 0$ satisfying $n \geq (K_1 + j + 1)\alpha$, which holds if $n+1 \geq K_1 + j + 1$ since $\frac{n}{n+1} \geq \alpha$. On the other hand, if $n+1 \leq K_1 + j$ we have $Df_{n+1}(j) \leq 0$ trivially. So $Df_{n+1}(j) \leq 0$ for all $j \geq 0$. Therefore, $Df_n(j) \leq 0$ and equivalently (7) holds for all $n \in \mathbb{N}, j \geq 0$.

The argument for (8) is similar. Let $k = K_2$ in (9). Note that $D^2 f_n(j) \geq 0$ is equivalent to (8) for $j \geq 0$, which trivially holds for $n \leq K_2 + j$ since $K_2 \geq K_1$ and $a_{K_2+j}^{(n)} - (j+1)a_{K_2+j+1}^{(n)} \geq 0$. In particular, (8) holds for $n = K_2, j \geq 0$. Assume that (8) holds for $n = m, j \geq 0$. We consider the case $n = m+1$. If $n = m+1 \leq K_2 + j$, (8) holds trivially. Otherwise, $m+1 \geq K_2 + j + 1$ and by applying (11) we see that $D^2 f_{n+1} \geq 0$. By induction, (8) holds for all $n \in \mathbb{N}, j \geq 0$. \square

Remark. The argument of the previous lemma shows that $(-1)^i D^i f_n(j) \geq 0$ for all $n \in \mathbb{N}, j \geq 0$ if we choose k so that $\frac{k}{k+i} \leq \alpha$.

For a fixed $K \geq K_1$, let

$$F_n(x) = x^{-K} \sum_{j=1}^n a_j^{(n)} x^j = \sum_{j=-\infty}^{\infty} a_{K+j}^{(n)} x^j. \quad (12)$$

and for a fixed $K \geq K_2$, let

$$G_n(x) = x^{-K} \sum_{j=0}^n (a_j^{(n)} - j a_{j+1}^{(n)}) x^j = \sum_{j=-\infty}^{\infty} (a_{K+j}^{(n)} - j a_{K+j+1}^{(n)}) x^j. \quad (13)$$

Lemma 3. *Let $f(x) = F_n(x)$, or $G_n(x)$ for the given suitable K . We have $(f(x)e^{-x})' \leq 0$ and $f(x+rx) \leq e^{rx}f(x)$ for all $r, x > 0$.*

Proof. It is easy to see that $f(x) - f'(x) \geq 0$ for $x > 0$ by Lemma 2. So $(f(x)e^{-x})' = (f' - f)e^{-x} \leq 0$ and hence $f(x+rx) \leq e^{rx}f(x)$ for $r > 0$. \square

We now come to the main result of this section.

Theorem 1. *Let $0 < \alpha$, $c < 1$, and $s \geq 0$ be fixed. Then*

- (i) $e^{-cst^\alpha} - c^{K_1} e^{-st^\alpha}$ is completely monotone in t .
- (ii) $K_1 e^{-st^\alpha} + st^\alpha e^{-st^\alpha}$ is completely monotone in t .
- (iii) $\frac{1}{c^{K_2}(1-c)} e^{-cst^\alpha} - st^\alpha e^{-st^\alpha}$ is completely monotone in t .
- (iv) $(\max\{\frac{jK_1}{c^{K_1}}, \frac{j}{c^{K_2}(1-c)}\})^j e^{-cst^\alpha} \pm s^j t^{j\alpha} e^{-st^\alpha}$ are completely monotone in t for any $j \in \mathbb{N}$.

Proof. By dilation, we may assume $s = 1$. We prove (i) first. Let $x = t^\alpha$ and F_n be as in 12,

$$\frac{d^n}{dt^n} e^{-t^\alpha} = (-1)^n t^{-n} \sum_{k=1}^n a_k^{(n)} x^k e^{-x} = (-1)^n t^{-n+K\alpha} e^{-x} F_n(x)$$

and

$$\frac{d^n}{dt^n} e^{-ct^\alpha} = (-1)^n t^{-n} \sum_{k=1}^n c^k a_k^{(n)} x^k e^{-x} e^{-rx} = (-1)^n t^{-n+K\alpha} c^K e^{-cx} F_n(cx). \quad (14)$$

Applying Lemma 2 and Lemma 3 to F_n gives us

$$\frac{\frac{d^n}{dt^n} e^{-ct^\alpha}}{\frac{d^n}{dt^n} e^{-t^\alpha}} \geq c^K,$$

for any $K \geq K_1$. This implies (i) since e^{-t^α} is completely monotone for any $0 < \alpha \leq 1$.

We now prove (ii). Let $g(s, t) = e^{-st^\alpha} s^{-K_1}$. Then $-\partial_s g(s, t)$, is the limit of the family of functions

$$\frac{1}{s^{K_1+1}(c-1)}(e^{-st^\alpha} - c^{-K_1} e^{-cst^\alpha})$$

as $c \rightarrow 1$, which are completely monotone in t by (i). So

$$K_1 e^{-st^\alpha} + st^\alpha e^{-st^\alpha} = -s^{K_1+1} \partial_s g(s, t)$$

is completely monotone in t .

For (iii), we denote by $f^{(n)}(t) = \partial_t^n f(t)$ and, for $K \geq K_2 \geq K_1$, write

$$\begin{aligned} (t^\alpha e^{-t^\alpha})^{(n)} + K(e^{-t^\alpha})^{(n)} &= -\frac{1}{\alpha} [t(e^{-t^\alpha})']^{(n)} + K(e^{-t^\alpha})^{(n)} \\ &= -\frac{1}{\alpha} [t(e^{-t^\alpha})^{(n+1)} + n(e^{-t^\alpha})^{(n)}] + K(e^{-t^\alpha})^{(n)} \\ &= \frac{(-1)^n t^{-n}}{\alpha} \left[\sum_{k=1}^{\infty} (a_k^{(n+1)} - (n - K\alpha) a_k^{(n)}) t^{k\alpha} e^{-t^\alpha} \right] \\ &= (-1)^n t^{-n} \left[\sum_{k=1}^{\infty} (a_{k-1}^{(n)} - (k - K) a_k^{(n)}) t^{k\alpha} e^{-t^\alpha} \right] \\ &= (-1)^n t^{-n+K\alpha} \left[\sum_{k=-\infty}^{\infty} (a_{K+k-1}^{(n)} - k a_{K+k}^{(n)}) t^{k\alpha} e^{-t^\alpha} \right] \\ &= (-1)^n t^{-n+K\alpha} x e^{-x} G_n(x) \end{aligned} \tag{15}$$

with $x = t^\alpha$ and $G_n(x)$ defined as in 13, which depends on K . Lemma 3 says that $G_n(x)e^{-x}$ decreases in x if $K \geq K_2$ and note that $G_n(x)e^{-x} = -(F_n(x)e^{-x})' \geq 0$. We have

$$\begin{aligned} x G_n(x) e^{-x} &\leq \frac{1}{(1-c)} \int_{cx}^x G_n(s) e^{-s} ds \\ &= \frac{1}{(1-c)} \int_{cx}^x -(F_n(s) e^{-s})' ds \\ &\leq \frac{1}{(1-c)} F_n(cx) e^{-cx}, \end{aligned}$$

for $0 < c < 1$. Combing this inequality with (14) and (15) we get

$$\frac{(-1)^n \frac{d^n}{dt^n} (t^\alpha e^{-t^\alpha} + K_2 e^{-t^\alpha})}{(-1)^n \frac{d^n}{dt^n} e^{-ct^\alpha}} \leq \frac{1}{c^{K_2}(1-c)}.$$

This proves (iii) since e^{-ct^α} and e^{-t^α} are completely monotone.

For (iv), let $f(t) = \max\{\frac{K_1}{c^{K_1}}, \frac{1}{c^{K_2}(1-c)}\} e^{-cst^\alpha}$, $g(t) = st^\alpha e^{-st^\alpha}$. By (i), (ii) and (iii) we have that both $f + g, f - g$ are completely monotone in t . Recall

that complete monotonicity is preserved by multiplication. Note that

$$\begin{aligned} f^{j+1} + g^{j+1} &= \frac{1}{2}[(f^j - g^j)(f - g) + (f^j + g^j)(f + g)] \\ f^{j+1} - g^{j+1} &= \frac{1}{2}[(f^j - g^j)(f + g) + (f^j + g^j)(f - g)]. \end{aligned}$$

We get, by induction, that $(\max\{\frac{K_1}{c^{K_1}}, \frac{1}{c^{K_2}(1-c)\alpha}\})^j e^{-jcs t^\alpha} - s^j t^{j\alpha} e^{-jst^\alpha}$ is completely monotone for any $s > 0$, which implies (iv). \square

We will apply Theorem 1 to pointwise estimates of $\phi_{s,\alpha}(\lambda)$. Let us first list a few basic properties of $\phi_{s,\alpha}$.

Lemma 4. *For any $s > 0, 0 < \alpha, \beta < 1$, we have*

$$\phi_{s,\frac{1}{2}}(\lambda) = \frac{1}{2\sqrt{\pi}} s e^{-\frac{s^2}{4\lambda}} \lambda^{-\frac{3}{2}}. \quad (16)$$

$$\phi_{1,\alpha\beta}(\lambda) = \int_0^\infty \phi_{s,\alpha}(\lambda) \phi_{1,\beta}(s) ds. \quad (17)$$

$$\phi_{s,\alpha}(\lambda) = s^{-\frac{1}{\alpha}} \phi_{1,\alpha}(s^{-\frac{1}{\alpha}} \lambda), \quad (18)$$

$$-\alpha s \partial_s \phi_{s,\alpha}(\lambda) = \phi_{s,\alpha}(\lambda) + \lambda \partial_\lambda \phi_{s,\alpha}(\lambda). \quad (19)$$

Proof. (16) is well-known (see e.g. [32], page 268). (17), (18) can be easily seen from (3) and (4). (18) implies (19). \square

Corollary 1. *For all $\lambda, s > 0, 0 < c < 1, j \in \mathbb{N}$, we have*

$$c^{K_1} \phi_{s,\alpha}(\lambda) \leq \phi_{cs,\alpha}(\lambda) \quad (20)$$

$$0 \leq \partial_\lambda (\lambda^{1+\alpha K_1} \phi_{s,\alpha}(\lambda)), \quad (21)$$

$$|s^j \partial_s^j \phi_{s,\alpha}(\lambda)| \leq \left(\max \left\{ \frac{j K_1}{c^{K_1}}, \frac{j}{c^{K_2}(1-c)\alpha} \right\} \right)^j \phi_{cs,\alpha}, \quad (22)$$

$$|s \partial_s \phi_{s,\alpha}(\lambda)| \leq \left(\frac{10}{1-\alpha} \right) \phi_{\alpha s,\alpha}(\lambda), \quad (23)$$

$$|s^j \partial_s^j \phi_{s,\alpha}(\lambda)| \leq \left(\frac{10j}{1-\alpha} \right)^j \phi_{\alpha s,\alpha}(\lambda). \quad (24)$$

Proof. These are direct consequences of Theorem 1, the identity (3), and the Hausdorff-Bernstein-Widder Theorem because $K_i \leq \frac{i}{1-\alpha}$, except that (21) requires a little more calculation. To prove (21), note that (5) and Theorem 1 (ii) imply that

$$\partial_s \frac{\phi_{s,\alpha}(\lambda)}{s^{K_1}} = -s^{-K_1-1} (K_1 \phi_{s,\alpha}(\lambda) - s \partial_s \phi_{s,\alpha}(\lambda)) \leq 0.$$

Since $\phi_{s,\alpha}(\lambda) = s^{-\frac{1}{\alpha}} \phi_{1,\alpha}(s^{-\frac{1}{\alpha}} \lambda)$, we get

$$-\left(\frac{1}{\alpha} + K_1 \right) s^{-\frac{1}{\alpha}-K_1-1} \phi_{1,\alpha}(s^{-\frac{1}{\alpha}} \lambda) - \frac{1}{\alpha} s^{-\frac{1}{\alpha}-1} \lambda s^{-\frac{1}{\alpha}-K_1} (\partial_\lambda \phi_{1,\alpha})(s^{-\frac{1}{\alpha}} \lambda) \leq 0.$$

That is

$$(1 + K_1\alpha)\phi_{1,\alpha}(s^{-\frac{1}{\alpha}}\lambda) + \lambda s^{-\frac{1}{\alpha}}(\partial_\lambda\phi_{1,\alpha})(s^{-\frac{1}{\alpha}}\lambda) \geq 0.$$

Therefore

$$(1 + K_1\alpha)\phi_{s,\alpha}(\lambda) + \lambda\partial_\lambda\phi_{s,\alpha}(\lambda) \geq 0,$$

since $\partial_\lambda\phi_{s,\alpha}(\lambda) = s^{-\frac{2}{\alpha}}\partial_\lambda\phi_{s,\alpha}(s^{-\frac{1}{\alpha}}\lambda)$. This is (21). \square

Lemma 5. *For any $s > 0, 0 < \beta < \alpha < 1$, we have that*

$$\int_0^\infty \left| \ln \left(s^{-\frac{1}{\alpha}} u \right) \right| \phi_{s,\alpha}(u) du < \frac{c}{\beta}. \quad (25)$$

$$\int_0^\infty \int_0^\infty \left| \ln \left(\frac{u}{v} \right) \right| \phi_{s,\alpha}(u) \phi_{s,\alpha}(v) dudv < \frac{c}{\beta^2}. \quad (26)$$

Proof. Since $\phi_{s,\alpha}(u) = s^{-\frac{1}{\alpha}}\phi_{1,\alpha}(s^{-\frac{1}{\alpha}}u)$, the left hand side of (25) is independent of s . We only need to prove the case $s = 1$. For $\alpha = \frac{1}{2}$, we can verify directly from (16) that (25) holds. Denote by $u(\alpha)$ the left hand side of (25). We then get $u(\frac{1}{2}) < \infty$. Using (17), we get $u(\frac{1}{2^n}) < \infty$. Now, for $\alpha > \frac{1}{2^n}$, we use (17) again and get

$$\begin{aligned} \phi_{1,\frac{1}{2^n}}(\lambda) &= \int_0^\infty \phi_{s,\alpha}(\lambda) \phi_{1,\frac{1}{\alpha 2^n}}(s) ds \\ &\geq \int_0^1 \phi_{s,\alpha}(\lambda) \phi_{1,\frac{1}{\alpha 2^n}}(s) ds \\ (by (20)) &\geq \phi_{1,\alpha}(\lambda) \int_0^1 s^{K_1(\alpha)} \phi_{1,\frac{1}{\alpha 2^n}}(s) ds \\ &\geq c_\alpha \phi_{1,\alpha}(\lambda). \end{aligned}$$

We conclude that $u(\alpha) < \infty$ for all $0 < \alpha < 1$. Since $\phi_{1,\alpha}(\lambda)$ is continuous as a function in α and this continuity is uniform for $\lambda \in [\delta, N]$ for any $0 < \delta < N < \infty$, one can easily see that $u(\alpha)$ is continuous in α for $\alpha \in (0, 1)$. We conclude that $u(\alpha)$ is bounded on $[\frac{1}{2^n}, \frac{1}{2}]$ for any $n \in \mathbb{N}$. Note that (17) also implies that

$$\begin{aligned} &\int_0^\infty \phi_{1,\alpha\beta}(\lambda) |\ln \lambda| d\lambda \\ &= \int_0^\infty \int_0^\infty \phi_{s,\alpha}(\lambda) |\ln \lambda| d\lambda \phi_{1,\beta}(s) ds \\ &= \int_0^\infty \int_0^\infty \phi_{1,\alpha}(v) |\ln(s^{\frac{1}{\alpha}}v)| dv \phi_{1,\beta}(s) ds \\ &\geq \pm \int_0^\infty \int_0^\infty \phi_{1,\alpha}(v) \left(\frac{1}{\alpha} |\ln s| - |\ln v| \right) dv \phi_{1,\beta}(s) ds \quad (27) \end{aligned}$$

$$\left(\leq \int_0^\infty \int_0^\infty \phi_{1,\alpha}(v) \left(\frac{1}{\alpha} |\ln s| + |\ln v| \right) dv \phi_{1,\beta}(s) ds \right) \quad (28)$$

Our change in the order of integration is justified because all the terms are positive. Note $\int_0^\infty \phi_{t,\alpha}(s)ds = 1$ for any t, α . (27) and (28) imply that

$$|u(\alpha) - \frac{1}{\alpha}u(\beta)| \leq u(\alpha\beta) \leq u(\alpha) + \frac{1}{\alpha}u(\beta) \quad (29)$$

We then obtain (25). (26) follows from (25). \square

Remark (Bell Polynomials). We define the complete Bell polynomial $B_n(x_1, \dots, x_n)$ by its generating function

$$\exp\left(\sum_{j=1}^{\infty} x_j \frac{u^j}{j!}\right) = \sum_{n=0}^{\infty} B_n(x_1, \dots, x_n) \frac{u^n}{n!}$$

From this, we get the formula

$$B_n(x_1, \dots, x_n) = \frac{d^n}{du^n} \exp\left(\sum_{j=1}^{\infty} x_j \frac{u^j}{j!}\right) \Big|_{u=0}$$

Now, for $s > 0$, let

$$x_j = -s \frac{d^j}{dt^j} t^\alpha = -s(\alpha)_j t^{\alpha-j}, \quad (30)$$

where $(\alpha)_j$ denotes the falling factorial. Then

$$\sum_{j=1}^{\infty} x_j \frac{u^j}{j!} = -st^\alpha \sum_{j=1}^{\infty} \frac{(\alpha)_j}{j!} \left(\frac{u}{t}\right)^j = st^\alpha - st^\alpha \left(1 + \frac{u}{t}\right)^\alpha = st^\alpha - s(t+u)^\alpha$$

Applying Theorem 1 part (i), we see that for all $n \in \mathbb{N}$, $c \in (0, 1)$, and $t > 0$ it holds that

$$\frac{\frac{d^n}{du^n} e^{-sc(t+u)^\alpha} \Big|_{u=0}}{\frac{d^n}{du^n} e^{-s(t+u)^\alpha} \Big|_{u=0}} \geq c^{K_1},$$

where K_1 is as in Lemma 2. We can rewrite this inequality as

$$e^{(1-c)st^\alpha} \frac{\frac{d^n}{du^n} e^{sct^\alpha - sc(t+u)^\alpha} \Big|_{u=0}}{\frac{d^n}{du^n} e^{st^\alpha - s(t+u)^\alpha} \Big|_{u=0}} \geq c^{K_1}.$$

We conclude that if we define x_j by (30), then

$$e^{(1-c)st^\alpha} \frac{B_n(cx_1, \dots, cx_n)}{B_n(x_1, \dots, x_n)} \geq c^{K_1} \quad (31)$$

for all $n \in \mathbb{N}$, $c \in (0, 1)$, and $t > 0$. All of these calculations are easily reversible, and we conclude that (31) is actually equivalent to part (i) of Theorem 1.

2 Positive semigroups and BMO

Let (M, σ, μ) be a sigma-finite measure space. Let $L^1(M)$ be the space of all complex valued integrable functions and $L^\infty(M)$ be the space of all complex valued measurable and essentially bounded functions on M . Denote by f^* the pointwise complex conjugate of a function f on M and by $\langle f, g \rangle$ the duality bracket $\int fg^*$.

Definition 1. A map T from $L^\infty(M)$ to $L^\infty(M)$ is called *positive* if $Tf \geq 0$ for $f \geq 0$. If T is positive on $L^\infty(M)$, then $T \otimes id$ is positive on matrix valued function spaces $L^\infty(M) \otimes M_n$ for all $n \in \mathbb{N}$, i.e. T is *completely positive*.

A positive map T commutes with complex conjugation, i.e. $T(f^*) = T(f)^*$. For two positive maps S, T , we will write $S \geq T$ if $S - T$ is positive.

We will need the following Kadison-Schwarz inequality for completely positive maps T ,

$$|T(f)|^2 \leq \|T(1)\|_{L^\infty} T(|f|^2), \quad f \in L^\infty(M). \quad (32)$$

2.1 Postive semigroups

We will consider a semigroup $(T_t)_{t \geq 0}$ of positive, weak*-continuous contractions on L^∞ with the weak* continuity at $t = 0+$. That is a family of positive, weak*-continuous contractions $T_t, t \geq 0$ on L^∞ such that $T_s T_t = T_{s+t}$, $T_0 = id$ and $\langle T_t(f), g \rangle \rightarrow \langle f, g \rangle$ as $t \rightarrow 0+$ for any $f \in L^\infty, g \in L^1$.

Such a semigroup (T_y) always admits an infinitesimal negative generator $L = \lim_{y \rightarrow 0} \frac{id - T_y}{y}$ which has a weak*-dense domain $D(L) \subset L^\infty$. We will write $T_y = e^{-yL}$. These definitions and facts extend to the noncommutative setting. Namely, given a semifinite von Neumann algebra \mathcal{M} and a normal semifinite faithful trace τ , we let $L^\infty(\mathcal{M}) = \mathcal{M}$ and $L^1(\mathcal{M})$ be the completion of $\{f \in \mathcal{M} : \|f\|_{L^1} = \tau|f| < \infty\}$. Here $|g| = (g^*g)^{\frac{1}{2}}$ and g^* denotes the adjoint operators of g and we set $\langle f, g \rangle = \tau(fg^*)$. We say a map T on \mathcal{M} is completely positive if $(T \otimes id)(f) \geq 0$ for any $f \geq 0, f \in \mathcal{M} \otimes M_n$. We say f_λ weak* converges to f if $\lim_\lambda \langle f_\lambda, g \rangle = \langle f, g \rangle$ for all $g \in L^1(\mathcal{M})$ (see [21] for details).

The so-called subordinated semigroups $T_{y,\alpha} = e^{-yL^\alpha}, 0 < \alpha < 1$ are defined as

$$T_{t,\alpha} f = \int_0^\infty T_u f \phi_{t,\alpha}(u) du = \int_0^\infty T_{t^{\frac{1}{\alpha}} u} f \phi_{1,\alpha}(u) du, \quad (33)$$

with $\phi_{t,\alpha}$ given in Section 1. The generator L^α is given by

$$L^\alpha(f) = \Gamma(-\alpha)^{-1} \int_0^\infty (T_t - id)(f) t^{-1-\alpha} dt, \quad (34)$$

for $f \in D(L)$. There are other (equivalent) formulations for L^α . The formula (34) is due to Balakrishnan (see [5] and [32, page 260]). For $T_t = e^{-tz} id$ with $Re(z) \geq 0$, $L^\alpha = z^\alpha$ with a chosen principal value so that $Re(z^\alpha) \geq 0$.

$(T_{y,\alpha})$ is again a semigroup of positive weak*-continuous contractions. The semigroup has an analytic extension and has the well-known norm estimate that

$$\sup_{y>0} \|y^k \partial_y^k T_{y,\alpha}\| < c_k. \quad (35)$$

What we wish is a pointwise estimate.

Note that (33) implies

$$\frac{T_{y,\frac{1}{2}}}{y}(f) \leq \frac{T_{t,\frac{1}{2}}}{t}(f) \quad \text{and} \quad |y^k \partial_y T_{y,\frac{1}{2}} f| \leq c_{k,t} T_{t,\frac{1}{2}} f, \quad (36)$$

for any $0 \leq t \leq y$ and $f \geq 0$ because of the positivity of T_u and the precise formulation of $\phi_{y,\frac{1}{2}}$.

Corollary 1 and the identity (33) actually imply the following corollary.

Corollary 2. *For all $f \geq 0$, $s > 0$, $0 < c, \alpha < 1$, and $j \in \mathbb{N}$, we have*

$$c^{K_1} T_{s,\alpha} f \leq T_{cs,\alpha} f \quad (37)$$

$$|s^j \partial_s^j T_{s,\alpha}(f)| \leq \left(\frac{10j}{1-\alpha}\right)^j T_{\alpha s,\alpha}(f). \quad (38)$$

Remark. When $\alpha = 1$, a similar estimate to Corollary 2 may hold for some special semigroups. For example, the heat semigroups generated by the Laplacian operator on \mathbb{R}^n has a similar estimate with $c > 1$. But one can not hope this in general since (38) is stronger than the analyticity on L^∞ .

2.2 Γ^2 criterion

P. A Meyer's gradient form Γ (also called ‘‘Carré du Champ’’) associated with T_t is defined as,

$$2\Gamma_L(f, g) = -L(f^*g) + (L(f^*)g) + f^*(L(g)), \quad (39)$$

for f, g with $f^*, g, f^*g \in D(L)$. It is easy to verify that for $L = -\Delta = -\frac{\partial^2}{\partial x^2}$, $\Gamma_L(f, g) = \nabla f^* \cdot \nabla g$.

Convention. We will write $\Gamma(f)$ for $\Gamma_L(f, f)$.

It is well known that the complete positivity of the operators T_t implies that $\Gamma(f, g)$ is a completely positive bilinear form. We then have the Cauchy-Schwartz inequality

$$\Gamma\left(\int_0^\infty a_s d\mu(s), \int_0^\infty a_s d\mu(s)\right) \leq \int_0^\infty d|\mu|(s) \int_0^\infty \Gamma(a_s, a_s) d|\mu|(s) \quad (40)$$

Bakry-Émery's Γ^2 criterion plays an important role in this article. We use an equivalent definition.

Definition 2. A semigroup of positive operator $(T_t)_t$ satisfies the $\Gamma^2 \geq 0$ criterion if $\Phi(s) = T_{s-u}|T_u f|^2$, $s > u$ is (midpoint) convex in u , i.e.

$$T_t|T_u f|^2 - |T_t T_u f|^2 \leq T_u(T_t|f|^2 - |T_t f|^2) \quad (41)$$

for all $t, u > 0$ and $f \in L^\infty$.

For L equal to the Laplace-Beltrami operator on a complete manifold, the $\Gamma^2 \geq 0$ criterion holds if the manifold has nonnegative Ricci curvature everywhere. The “ Γ^2 ” criterion is satisfied by a large class of semigroups including the heat, Ornstein-Uhlenbeck, Laguerre, and Jacobi semigroups (see [2]), and also by the semigroups of completely positive contractions on group von Neumann algebras. We refer the reader to [3] and references therein for the so-called curvature-dimension criterion which is more general than the “ Γ^2 ” criterion.

D. Bakry usually assumes that there exists a $*$ -algebra \mathcal{A} which is weak* dense in $L^\infty(M)$ such that $T_s(\mathcal{A}) \subset \mathcal{A} \subset D(L)$. This is not needed in this article because we will only use the form $T_{t,\alpha}\Gamma_{L^\beta}(T_{s,\alpha}f, T_{s,\alpha}g)$, $0 < \alpha < 1$, $\alpha \leq \beta \leq 1$ which is well defined as

$$-L^\beta T_{t,\alpha}[(T_{s,\alpha}f^*)(T_{s,\alpha}g)] + T_{t,\alpha}[(T_{s,\alpha}f^*)(L^\beta T_{s,\alpha}g)] + T_{t,\alpha}[(L^\beta T_{s,\alpha}f^*)(T_{s,\alpha}g)] \quad (42)$$

for all $f, g \in L^\infty$ since $T_{s,\alpha}(L^\infty) \subset D(L) \subset D(L^\alpha)$ because of (33).

We will need the following Lemma due to P.A. Meyer. We add a short proof for the convenience of the reader.

Lemma 6. For any $f \in L^\infty$ such that $T_s f, T_s f^*, T_s |f|^2 \in D(L)$ for all $s > 0$, we have

$$T_s |f|^2 - |T_s f|^2 = 2 \int_0^s T_{s-t} \Gamma(T_t f) dt.$$

In particular, for $0 < \alpha < 1$,

$$T_{s,\alpha} |f|^2 - |T_{s,\alpha} f|^2 = 2 \int_0^s T_{s-t,\alpha} \Gamma_{L^\alpha}(T_{t,\alpha} f) dt \quad (43)$$

for any $f \in L^\infty$.

Proof. For s fixed, let

$$F_t = T_{s-t}(|T_t f|^2).$$

Then

$$\begin{aligned} \frac{\partial F_t}{\partial t} &= \frac{\partial T_{s-t}}{\partial t}(|T_t f|^2) + T_{s-t}[(\frac{\partial T_t}{\partial t} f^*)f] + T_{s-t}[f^*(\frac{\partial T_t}{\partial t} f)] \\ &= -2T_{s-t}\Gamma(T_t f). \end{aligned} \quad (44)$$

Therefore

$$T_s |f|^2 - |T_s f|^2 = -F_s + F_0 = 2 \int_0^s T_{s-t} \Gamma(T_t f) dt.$$

Since $T_{s,\alpha}(L^\infty) \subset D(L^\alpha)$ we get (43) for all $f \in L^\infty$. \square

Remark. Equation (44) shows that the $\Gamma^2 \geq 0$ criterion implies that

$$T_s \Gamma(T_{v+t} f) \leq T_{v+s}(\Gamma(T_t f)) \quad (45)$$

for all $v, s, t > 0$ and $f \in L^\infty$ such that $T_s f, T_s f^*, T_s |f|^2 \in D(L)$ for all $s > 0$.

The following lemma says that the $\Gamma^2 \geq 0$ criterion passes to fractional powers, which could be known to some experts. We add a proof as we do not find a reference.

Lemma 7. *If $T_t = e^{-tL}$ satisfies the $\Gamma^2 \geq 0$ criterion (41), then $T_{t,\alpha} = e^{-tL^\alpha}$ satisfies (41) and (45) for all $f \in L^\infty$ and $0 < \alpha < 1$. Moreover,*

$$\Gamma_{L^\alpha}(s^j \partial_s^j T_{s,\alpha} f) \leq \left(\frac{10}{1-\alpha} \right)^j T_{s,\alpha} \Gamma_{L^\alpha}(f) \quad (46)$$

Proof. Applying (34), we have that, with $c_\alpha = -(\Gamma(-\alpha))^{-1} > 0$,

$$\begin{aligned} \Gamma_{L^\alpha}(f, f) &= c_\alpha \int_0^\infty (T_t |f|^2 - (T_t f^*) f - f^* (T_t f) + |f|^2) t^{-1-\alpha} dt \\ &= c_\alpha \int_0^\infty (T_t |f|^2 - |T_t f|^2 + |T_t f - f|^2) t^{-1-\alpha} dt. \end{aligned} \quad (47)$$

if $f, f^*, |f|^2 \in D(L)$. The integration converges because

$$\|T_t |f|^2 - |T_t f|^2\| \leq c \min\{t, 1\}, \quad (48)$$

for $f \in D(L)$. In fact, by the $\Gamma^2 \geq 0$ criterion (41), we see that

$$T_t |T_t f|^2 - |T_{2t} f|^2 \leq \frac{1}{2} (T_{2t} |f|^2 - |T_{2t} f|^2).$$

So

$$\begin{aligned} \|T_t |f|^2 - |T_t f|^2\|^\frac{1}{2} &\leq \|T_t |f - T_t f|^2 - |T_t (f - T_t f)|^2\|^\frac{1}{2} + \|T_t |T_t f|^2 - |T_{2t} f|^2\|^\frac{1}{2} \\ &\leq ct + 2^{-\frac{1}{2}} \|T_{2t} |f|^2 - |T_{2t} f|^2\|^\frac{1}{2}. \end{aligned}$$

Let $u(t) = t^{-\frac{1}{2}} \|T_t |f|^2 - |T_t f|^2\|^\frac{1}{2}$. We get

$$u(t) \leq ct^\frac{1}{2} + u(2t).$$

Since $u(t)$ is uniformly bounded on $[1, \infty)$, we get $u(t)$ is uniformly bounded on $[0, \infty)$ by iteration. This proves (48). Therefore, by the Cauchy-Schwartz inequality (40) and the $\Gamma^2 \geq 0$ criterion for T_t we get

$$\Gamma_{L^\alpha}(T_u f, T_u f) \leq T_u \Gamma_{L^\alpha}(f, f). \quad (49)$$

Applying the subordination formula that $T_{t,\alpha} = \int_0^\infty T_u \phi_{t,\alpha}(u) du$ and the Cauchy-Schwartz inequality (40), we obtain

$$\Gamma_{L^\alpha}(T_{t,\alpha} f, T_{t,\alpha} f) \leq T_{t,\alpha} \Gamma_{L^\alpha}(f, f). \quad (50)$$

One can easily adapt the proof to get

$$T_{u,\alpha}\Gamma_{L^\alpha}(T_{t,\alpha}T_{v,\alpha}g, T_{t,\alpha}T_{v,\alpha}g) \leq T_{u,\alpha}T_{t,\alpha}\Gamma_{L^\alpha}(T_{v,\alpha}g, T_{v,\alpha}g). \quad (51)$$

for all $g \in L^\infty$ since $T_{v,\alpha}g, T_{u,\alpha}|T_{v,\alpha}g|^2 \in D(L)$. Applying (43), we get (45) for $T_{t,\alpha}$.

Now, apply (40) to Γ_{L^α} and $a(s) = T_s f, d\mu(s) = s^j \partial_j \phi_{t,\alpha}(s) ds$; we get (46) from (33), (24), and (51). \square

2.3 BMO spaces associated with semigroups of operators

BMO spaces associated with semigroup generators have been intensively studied recently (see [12]). In this article, we follow the ones studied in [20] and [24] because they are defined in a pure semigroup language. Set

$$\|f\|_{\text{bmo}(L^\alpha)} = \sup_{0 < t < \infty} \|T_{t,\alpha}|f|^2 - |T_{t,\alpha}f|^2\|_{L^\infty}^{\frac{1}{2}}, \quad (52)$$

$$\|f\|_{\text{BMO}(L^\alpha)} = \sup_{0 < t < \infty} \|T_{t,\alpha}|f - T_{t,\alpha}f|^2\|_{L^\infty}^{\frac{1}{2}}, \quad (53)$$

for $f \in L^\infty, 0 < \alpha \leq 1$.

We wish to define the space $\text{BMO}(L^\alpha), 0 < \alpha \leq 1$ so that it is a dual space and L_0^∞ is weak* dense in it, to be consistent with the classical ones (where $L_0^\infty(M) = L^\infty(M)/\ker L^\alpha$). In [20] and [24], this is done by using a SOT-topology in the corresponding Hilbert C^* modulars. In this article, we prefer to use the following detour to avoid introducing the theory of Hilbert C^* modulars. Define, for $g \in L^1$,

$$\|g\|_{H^1(L^\alpha)} = \sup\{|\langle f, g \rangle| : f \in L^\infty, \|f\|_{\text{BMO}(L^\alpha)}, \|f^*\|_{\text{BMO}(L^\alpha)} \leq 1\}. \quad (54)$$

Let $H^1(L^\alpha) = \{g \in L^1; \|g\|_{H^1} < \infty\}$. For a net $f_\lambda \in L_0^\infty(M)$, we say f_λ converges in the *weak* topology* if $\langle f_\lambda, g \rangle$ converges for any $g \in H^1(L^\alpha)$. Let $\text{BMO}(L^\alpha)$ be the abstract closure of $L_0^\infty(M)$ with respect to this weak* topology, that is the linear space of all weak* convergent nets $f_\lambda \in L_0^\infty(M)$. For a weak* convergent f_λ , let

$$\|\lim_\lambda f_\lambda\|_{\text{BMO}(L^\alpha)} = \sup_{\|g\|_{H^1} \leq 1} \lim_\lambda \langle f_\lambda, g \rangle.$$

It is easy to see that this coincides with (53) if $\lim_\lambda f_\lambda \in L^\infty$.

As an application of Corollary 2, we show that these BMO and bmo norms with different $0 < \alpha < 1$ are all equivalent if we assume the $\Gamma^2 \geq 0$ criterion.

Lemma 8. *Suppose L generates a weak* continuous semigroup of positive contractions, we have*

$$\|f\|_{\text{BMO}(L^\beta)} \leq \frac{c\alpha}{\beta} \|f\|_{\text{BMO}(L^\alpha)}, \quad (55)$$

$$\|f\|_{\text{BMO}(L^\beta)} \leq \frac{4}{1-\beta} \|f\|_{\text{bmo}(L^\beta)}, \quad (56)$$

for any $0 < \beta < \alpha \leq 1$. Assuming in addition that the semigroup $T_t = e^{-tL}$ satisfies the $\Gamma^2 \geq 0$ criterion (45), we have that

$$\|f\|_{BMO(L^\alpha)} \simeq \|f\|_{bmo(L^\alpha)} \simeq \|f\|_{bmo(L^\beta)}, \quad (57)$$

for all $0 < \beta, \alpha < 1$. In particular,

$$c(1 - \alpha)^2 \|f\|_{BMO(L^\alpha)} \leq \|f\|_{BMO(\sqrt{L})} \leq c \|f\|_{BMO(L^\alpha)}, \quad (58)$$

for all $\frac{1}{2} < \alpha < 1$.

Proof. The argument for (55) is the same as that for the second inequality of [20, Theorem 2.6]. We sketch it here. By the Cauchy-Schwartz inequality,

$$\begin{aligned} T_{t,\beta}|f - T_{t,\beta}f|^2 &= \int_0^\infty \phi_{t,\frac{\beta}{\alpha}}(u) T_{u,\alpha} \left| \int_0^\infty \phi_{t,\frac{\beta}{\alpha}}(v) (f - T_{v,\alpha}f) dv \right|^2 du \\ &\leq \int_0^\infty \int_0^\infty \phi_{t,\frac{\beta}{\alpha}}(u) \phi_{t,\frac{\beta}{\alpha}}(v) T_{u,\alpha} |f - T_{v,\alpha}f|^2 dudv. \end{aligned}$$

It is easy to see that $\|T_{u,\alpha}|f - T_{v,\alpha}f|^2\| \leq (1 + |\ln \frac{u}{v}|) \|f\|_{BMO(L^\alpha)}^2$, so we get (55) from (26).

For the rest of this proof, we use Γ for Γ_{L^β} , T_t for $T_{t,\beta}$ and P_t for $T_{t,\frac{\beta}{2}}$ to simplify the notation. Since T_t has the quasi monotone property (37), we have

$$P_t = \int_0^\infty T_u \phi_{t,\frac{1}{2}}(u) du \geq \int_0^{t^2} \left(\frac{u}{t^2}\right)^{K_1} T_{t^2} \phi_{t,\frac{1}{2}}(u) du \geq \frac{1}{100K_1} T_{t^2}. \quad (59)$$

We now prove (56). Note

$$\begin{aligned} \|T_t|f - T_t f|^2\| &= \|T_t|f - T_t f|^2 - |T_t f - T_t T_t f|^2 + |T_t f - T_t T_t f|^2\| \\ &\leq \|f - T_t f\|_{bmo(L^\beta)}^2 + \|T_t f - T_{2t} f\|^2. \end{aligned}$$

Let $\gamma = 2^{\frac{1}{K_1}}$ and $S = 2T_t - T_{\gamma t}$. Then S is a unital completely positive map because of (37). We have

$$\begin{aligned} |T_t f - T_{\gamma t} f|^2 + |S f - T_t f|^2 &= -2|T_t f|^2 + |T_{\gamma t} f|^2 + |S f|^2 \\ &\leq -2|T_t f|^2 + T_{\gamma t}|f|^2 + S|f|^2 \\ &\leq -2|T_t f|^2 + 2T_t|f|^2 \\ &\leq 2\|f\|_{bmo(L^\beta)}^2. \end{aligned}$$

We get by the triangle inequality that

$$\|T_t f - T_{2t} f\| \leq K_1 \sup_s \|T_s f - T_{\gamma s} f\| \leq \sqrt{2} K_1 \|f\|_{bmo(L^\beta)}.$$

Therefore,

$$\|f\|_{BMO(L^\beta)} \leq \sqrt{4 + 2K_1^2} \|f\|_{bmo(L^\beta)}.$$

To prove (57), we note that the $\Gamma^2 \geq 0$ assumption for L passes to L^α by Lemma 7. The inequality $\|f\|_{bmo} \leq (2 + \sqrt{2})\|f\|_{BMO}$ is proved in [20, Proposition 2.4] assuming the $\Gamma^2 \geq 0$ criterion. Together with (56), we get $\|f\|_{BMO(L^\alpha)} \simeq \|f\|_{bmo(L^\alpha)}$. We now show the second equivalence in (57). Note,

$$\begin{aligned}
\int_0^t T_{t-s} \Gamma(T_s P_{\sqrt{t}} f) ds &= \int_0^t T_{t-s} \Gamma \left(\int_0^\infty \phi_{\sqrt{t}, \frac{1}{2}}(v) T_v T_s f dv \right) ds \\
&\leq \int_0^\infty \phi_{\sqrt{t}, \frac{1}{2}}(v) \int_0^t T_{t-s} \Gamma(T_v T_s f) ds dv \\
&\leq \int_0^\infty \phi_{\sqrt{t}, \frac{1}{2}}(v) \int_0^t T_{t+v-\frac{t+v}{t}s} \Gamma(T_{\frac{t+v}{t}s} f) ds dv \\
(u = \frac{t+v}{t}s) &\leq \int_0^\infty \phi_{\sqrt{t}, \frac{1}{2}}(v) \frac{t}{t+v} \int_0^{t+v} T_{t+v-u} \Gamma(T_u f) ds dv \\
(43) &= \int_0^\infty \phi_{\sqrt{t}, \frac{1}{2}}(v) \frac{t}{t+v} (T_{t+v}|f|^2 - |T_{t+v}f|^2) dv \\
&\leq \int_0^\infty \phi_{\sqrt{t}, \frac{1}{2}}(v) \frac{t}{t+v} \|f\|_{bmo(L^\beta)}^2 dv < \frac{5}{6} \|f\|_{bmo(L^\beta)}^2.
\end{aligned}$$

We then have

$$\begin{aligned}
&(T_t|f|^2 - |T_t f|^2)^{\frac{1}{2}} \\
&\leq (T_t|f - P_{\sqrt{t}}f|^2 - |T_t f - T_t P_{\sqrt{t}}f|^2)^{\frac{1}{2}} + (T_t|P_{\sqrt{t}}f|^2 - |T_t P_{\sqrt{t}}f|^2)^{\frac{1}{2}} \\
&\leq 100K_1(P_{\sqrt{t}}|f - P_{\sqrt{t}}f|^2)^{\frac{1}{2}} + \sqrt{\frac{5}{6}}\|f\|_{bmo(L^\beta)} \\
&\leq 100K_1\|f\|_{bmo(L^{\frac{\beta}{2}})} + \sqrt{\frac{5}{6}}\|f\|_{bmo(L^\beta)},
\end{aligned}$$

so

$$\|f\|_{bmo(L^\beta)} \leq 1200K_1\|f\|_{bmo(L^{\frac{\beta}{2}})}.$$

Therefore,

$$\|f\|_{BMO(L^\beta)} \leq 10000K_1^2\|f\|_{BMO(L^{\frac{\beta}{2}})}.$$

Applying (55), we have $\|f\|_{BMO(L^\alpha)} \simeq \|f\|_{BMO(L^\beta)}$ for all $0 < \beta, \alpha < 1$. \square

Remark. The equivalence (57) fails for $\alpha = 1$ in general. See Section 4, Example 2.

3 Imaginary powers and H^∞ -calculus

3.1 H^∞ -calculus.

Let us review some definitions and basic facts about H^∞ -calculus. We refer the readers to [8, 21, 14] for details. For $0 < \theta < \pi$, let S_θ be the following open

sector of the complex plane:

$$S_\theta = \{z \in \mathbb{C}, |\arg z| < \theta\}.$$

Recall that we say a closed operator A on a Banach space X is a *sectorial* operator of type $\omega < \pi$ if the spectrum of A is contained in $\overline{S_\omega}$, the closure of S_ω , and for any $\theta, \omega < \theta < \pi, z \notin S_\theta$, there exists c_θ such that

$$\|z(z - A)^{-1}\| \leq c_\theta.$$

We will assume that the domain of A is dense in X (or weak* dense in X when X is a dual space). We may also assume that A has dense range and is one to one by considering $A + \varepsilon$ (see [21, Lemma 3.2, 3.5]).

Let $H^\infty(S_\eta)$ be the space of all bounded analytic functions on S_η and $H_0^\infty(S_\eta)$ be the subspace of the functions $\Phi \in H^\infty(S_\eta)$ with an extra decay property that

$$|\Phi(z)| \leq \frac{c|z|^r}{(1 + |z|)^{2r}},$$

for some $c, r > 0$. Then for any $\Phi \in H_0^\infty(S_\eta)$, and $\theta > \eta$,

$$\Phi(A) = \frac{1}{2\pi i} \int_{\gamma_\theta} \Phi(z)(z - A)^{-1} dz \quad (60)$$

is a well defined bounded operator on $D(A)$ and its (weak*) extension is bounded on X . Here γ_θ is the boundary of S_θ oriented counterclockwise. For general $\Phi \in H^\infty(S_\eta)$, set

$$\Phi(A) = \psi(A)^{-1}(\Phi\psi)(A), \quad (61)$$

with $\psi(z) = \frac{z}{(1+z)^2}$. It turns out that the so defined $\Phi(A)$ is a closed (weak*) densely defined operator, which may not be bounded, and it coincides with $\Phi(A)$ defined as in (60) for $\Phi \in H_0^\infty(S_\eta)$. Moreover, these definitions are consistent with the definitions in the “older” functional calculus.

Definition 3. We say a (weak*) densely defined sectorial operator A of type ω has bounded $H^\infty(S_\eta)$ -calculus, $\omega < \eta < \pi$, if the map $\Phi(A)$ extends to a bounded operator on X and there is a constant C such that

$$\|\Phi(A)\| \leq C\|\Phi\|_{H^\infty(S_\eta)} \quad (62)$$

for any bounded analytic function $\Phi \in H^\infty(S_\eta)$.

Remark. Suppose a densely defined sectorial A has bounded $H^\infty(S_\eta)$ -calculus on Y and suppose Y is a weak* dense subspace of a dual Banach space X . Then the weak* extension of $\Phi(A)$ onto X , still denoted by $\Phi(A)$, is bounded and satisfies (62) with the same constant. So a weak* dense sectorial operator A has H^∞ -calculus on X if and only if it has H^∞ -calculus on the norm closure of $D(A)$.

The negative infinitesimal generator L of any uniformly bounded (weak*) strong continuous semigroup on a dual Banach space X is actually a (weak*) densely defined sectorial operator of type $\frac{\pi}{2}$ and L^α is of type $\frac{\alpha\pi}{2}$ on X . Cowling, Duong, and Hiebe & Prüss ([7, 11, 16]) prove that the negative infinitesimal generator of a semigroup of positive contractions on L^p , $1 < p < \infty$ always has the bounded $H^\infty(S_\eta)$ -calculus for any $\eta > \frac{\pi}{2}$. One cannot hope to extend this to $p = \infty$. We will prove that the associated $\text{BMO}(\sqrt{L})$ space is a good alternative, as desired.

Lemma 9. *Suppose A is a densely defined sectorial operator of type $\omega < \pi/2$ on a Banach space X . Assume $\int_0^\infty Ae^{-tA}a(t)dt$ is bounded on X with norm smaller than C for any function $a(t)$ with values in ± 1 . Then A has a bound $H^\infty(S_\eta^0)$ calculus for any $\eta > \pi/2$.*

Proof. This is a consequence of [8, Example 4.8] by setting $a(t)$ to be the sign of $\langle Te^{-tT}u, v \rangle$ for any pair (u, v) in a dual pair (X, Y) . \square

We are going to prove that the negative generator L of a semigroup of positive contractions satisfies the assumptions of Lemma 9. We follow an idea of E. Stein and consider scalar valued functions $a(t)$ such that

$$s \int_s^\infty \frac{|a(v-s)|^2}{v^2} dv \leq c_a^2, \quad (63)$$

for all $s > 0$ and some constant c_a . Define M_a by

$$M_a(f) = \int_0^\infty a(t) \frac{\partial T_{t,\alpha} f}{\partial t} dt = \int_0^\infty a(t) L^\alpha T_{t,\alpha} f dt, \quad (64)$$

for $f \in L^\infty$, $0 < \alpha < 1$. For now, we assume a is supported on a compact subset of $(0, \infty)$ so we do not worry about the convergence of the integration.

Lemma 10. *Assume that L generates a weak* continuous semigroup of positive contractions on L^∞ satisfying the $\Gamma^2 \geq 0$ criterion (45). We have*

$$\|M_a(f)\|_{\text{bmo}(L^\alpha)} \leq \frac{cc_a}{(1-\alpha)^2} \|f\|_{\text{bmo}(L^\alpha)}, \quad (65)$$

$$\|M_a(f)\|_{\text{BMO}(L^\alpha)} \leq \frac{cc_a}{(1-\alpha)^2} \|f\|_{L^\infty}, \quad (66)$$

for any $f \in L^\infty$, $0 < \alpha < 1$.

Proof. We consider the case $\alpha \geq \frac{3}{4}$ only. The case $\alpha < \frac{3}{4}$ is easier and follows from this case by subordination. Recall that the $\Gamma^2 \geq 0$ assumption for L passes to L^α by Lemma 7, and $T_{t,\alpha}(L^\infty) \subset D(L^{2\alpha})$, $L^{2\alpha}T_{t,\alpha} = \partial_t^2 T_{t,\alpha}$. In this proof, we use Γ for Γ_{L^α} the gradient form associated with L^α , T_t for $T_{t,\alpha}$ and P_t for

$T_{t, \frac{\alpha}{2}}$ to simplify the notation. Let $r = \frac{1}{1-\alpha} > 4$. We have that

$$\begin{aligned}
\int_0^t T_{rt-s} \Gamma(T_s f) ds &= \int_0^t T_{rt-s} \Gamma \left(\int_s^\infty L^\alpha T_v f dv \right) ds \\
&\leq \int_0^t T_{rt-s} \int_s^\infty \Gamma(L^\alpha T_v f) v^{\frac{3}{2}} dv \int_s^\infty v^{-\frac{3}{2}} dv ds \\
&= \int_0^t T_{rt-s} \int_s^\infty \Gamma(L^\alpha T_v f) v^{\frac{3}{2}} dv 2s^{-\frac{1}{2}} ds \\
&= \int_0^\infty \int_0^{t \wedge v} 2s^{-\frac{1}{2}} T_{rt-s} ds \Gamma(L^\alpha T_v f) v^{\frac{3}{2}} dv
\end{aligned}$$

Let $S_v = \int_0^{t \wedge v} 2s^{-\frac{1}{2}} T_{rt-s} ds$. So by Lemma 6 and the $\Gamma^2 \geq 0$ criterion,

$$\begin{aligned}
\|f\|_{bmo}^2 &= \sup_t \left\| \int_0^{rt} T_{rt-s} \Gamma(T_s f) ds \right\| \\
&\leq \left\| \sup_t \int_0^{rt} T_{rt-\frac{s}{r}} \Gamma(T_{\frac{s}{r}} f) ds \right\| \\
&= \left\| \sup_t r \int_0^t T_{rt-s} \Gamma(T_s f) ds \right\| \\
&\leq \sup_t r \left\| \int_0^\infty S_v \Gamma(L^\alpha T_v) v^{\frac{3}{2}} dv \right\|.
\end{aligned}$$

So,

$$\begin{aligned}
\frac{1}{r} \|M_a f\|_{bmo}^2 &\leq \left\| \int_0^\infty S_v \Gamma(L^\alpha T_v M_a(f)) v^{\frac{3}{2}} dv \right\| \\
&= \left\| \int_0^\infty S_v \Gamma(T_v \int_0^\infty a(u) L^{2\alpha} T_u f du) v^{\frac{3}{2}} dv \right\| \\
&= \left\| \int_0^\infty S_v \Gamma \left(\int_0^\infty a(u) L^{2\alpha} T_{u+v} f du \right) v^{\frac{3}{2}} dv \right\| \\
&= \left\| \int_0^\infty v^{\frac{3}{2}} S_v \Gamma \left(\int_v^\infty a(u-v) \frac{1}{u} u L^{2\alpha} T_u f du \right) dv \right\| \\
(\text{Inequality (40)}) &\leq \left\| \int_0^\infty v^{\frac{3}{2}} S_v \left(\int_v^\infty \frac{|a|^2}{u^2} du \int_v^\infty \Gamma(u L^{2\alpha} T_u f) du \right) dv \right\| \\
&\leq c_a^2 \left\| \int_0^\infty S_v \left(\int_v^\infty \Gamma(u L^{2\alpha} T_u f) du \right) v^{\frac{1}{2}} dv \right\| \\
&= c_a^2 \left\| \int_0^\infty \int_0^{u \wedge t} v^{\frac{1}{2}} S_v dv \Gamma(u L^{2\alpha} T_u f) du \right\|.
\end{aligned}$$

Note $K_1 \leq r$ and $\sup_{r>4} (\frac{2}{1+\alpha} \frac{r}{r-1})^r \leq c$. By (37), we have, for $u \leq t$,

$$\begin{aligned}
\int_0^{t \wedge u} v^{\frac{1}{2}} S_v dv &\leq \int_0^{t \wedge u} v^{\frac{1}{2}} \int_0^{t \wedge v} s^{-\frac{1}{2}} T_{\frac{1+\alpha}{2}(rt-u)} \left(\frac{2}{1+\alpha} \cdot \frac{r}{r-1} \right)^r ds dv \\
&\leq c T_{\frac{1+\alpha}{2}(rt-u)} t^2 \wedge u^2.
\end{aligned}$$

Applying (46), we get

$$\begin{aligned}
\left\| \int_0^t \int_0^{u \wedge t} v^{\frac{1}{2}} S_v dv \Gamma(u L^{2\alpha} T_{\frac{u}{2}} T_{\frac{u}{2}} f) du \right\| &\leq cr^2 \left\| \int_0^t \frac{T_{\frac{\alpha u}{2}}}{u^2} \int_0^{u \wedge t} v^{\frac{1}{2}} S_v dv \Gamma(T_{\frac{u}{2}} f) du \right\| \\
&\leq cr^2 \left\| \int_0^t T_{\frac{(1+\alpha)r}{2} - \frac{u}{2}} \Gamma(T_{\frac{u}{2}} f) du \right\| \\
&\leq cr^2 \left\| \int_0^{\frac{t}{2}} T_{\frac{t}{2}-s} \Gamma(T_s f) ds \right\| \leq cr^2 \|f\|_{bmo}^2.
\end{aligned}$$

For $\alpha^{-n}t < u \leq \alpha^{-n-1}t, n \geq 0$, we use

$$\int_0^{t \wedge u} v^{\frac{1}{2}} S_v dv \leq \int_0^{t \wedge u} v^{\frac{1}{2}} \int_0^{t \wedge v} 2s^{-\frac{1}{2}} T_{rt-t} \left(\frac{r}{r-1} \right)^r ds dv \leq c T_{rt-t} t^2 \wedge u^2.$$

Similar to (46), we get $\Gamma(u^2 L^{2\alpha} T_{\alpha^{-n}t} f) \leq cr^2 T_{2\alpha^{-n}t-u} \Gamma(f)$ because $\frac{r-1}{r-2} = \frac{1}{2-\alpha^{-1}} \leq \frac{\alpha^{-n}t}{2\alpha^{-n}t-u} \leq 1$. So

$$|\Gamma(u L^{2\alpha} T_{\alpha^{-n}t} T_{u-\alpha^{-n}t} f)| \leq c \frac{r^2}{u^2} T_{2\alpha^{-n}t-u} \Gamma(T_{u-\alpha^{-n}t} f)$$

Therefore,

$$\begin{aligned}
&\left\| \int_{\alpha^{-n}t}^{\alpha^{-n-1}t} \int_0^{u \wedge t} v^{\frac{1}{2}} S_v dv \Gamma(u L^{2\alpha} T_{\frac{u}{2}} T_{\frac{u}{2}} f) du \right\| \\
&\leq cr^2 \left\| \int_{\alpha^{-n}t}^{\alpha^{-n-1}t} \frac{t^2 \wedge u^2}{u^2} T_{2\alpha^{-n}t-u} \Gamma(T_{u-\alpha^{-n}t} f) du \right\| \\
&= cr^2 \alpha^{2n} \left\| \int_0^{\alpha^{-n-1}t(1-\alpha)} T_{\alpha^{-n}t-s} \Gamma(T_s f) ds \right\| \\
&\leq cr^2 \alpha^{2n} \|f\|_{bmo}^2.
\end{aligned}$$

Summing up for $n \geq 0$, we get

$$\left\| \int_t^\infty \int_0^{u \wedge t} v^{\frac{1}{2}} S_v dv \Gamma(u L^{2\alpha} T_{\frac{u}{2}} T_{\frac{u}{2}} f) du \right\| \leq cr^3 \|f\|_{bmo}^2.$$

Combining the estimates above, we conclude that

$$\|M_a(f)\|_{bmo(L^\alpha)} \leq cc_a r^2 \|f\|_{bmo(L^\alpha)}.$$

Applying (57), we actually get

$$\|M_a(f)\|_{BMO(L^\alpha)} \leq cc_a r^3 \|f\|_{BMO(L^\alpha)} \leq cc_a r^3 \|f\|_{L^\infty}.$$

But we wish to get a better estimate. Note

$$\begin{aligned}
(T_t - T_{2t})M_a(f) &= \int_0^\infty a(s)\partial_s(T_{t+s} - T_{2t+s})f ds \\
&= \int_t^\infty a(s-t)\partial_s(T_s - T_{t+s})f ds \\
&\leq \left(\int_t^\infty \frac{|a(s-t)|^2}{s^2} ds \right)^{\frac{1}{2}} \left(\int_t^\infty s^2 \left| \int_0^t \partial_s^2 T_{v+s} f dv \right|^2 ds \right)^{\frac{1}{2}} \\
&\leq c_a \left(\int_t^\infty s^2 \int_0^t |\partial_s^2 T_{v+s} f|^2 dv ds \right)^{\frac{1}{2}} \\
\text{(by (38)) } &\leq \frac{25c_a}{(1-\alpha)^2} \left(\int_t^\infty s^{-2} \int_0^t T_{\alpha(v+s)} |f|^2 dv ds \right)^{\frac{1}{2}}.
\end{aligned}$$

Therefore

$$\|(T_t - T_{2t})M_a(f)\|_{L^\infty} \leq \frac{25c_a}{(1-\alpha)^2} \|f\|_{L^\infty},$$

and hence

$$\|M_a(f)\|_{BMO(L^\alpha)} \leq \|M_a f\|_{bmo(L^\alpha)} + \sup_t \|(T_t - T_{2t})M_a(f)\|_{L^\infty} \leq cr^2 c_a \|f\|_{L^\infty}.$$

□

Given $f \in L^\infty, g \in H^1(L^\alpha)$, let $\tilde{a}(t) = \text{sign}\langle L^\alpha T_{t,\alpha} f, g \rangle a(t)$. Then \tilde{a} satisfies (63) if a does. We have from Lemma 10 that

$$\begin{aligned}
\int_0^\infty |\langle a(t) L^\alpha T_{t,\alpha} f, g \rangle| dt &= \lim_{N, M \rightarrow \infty} \int_{\frac{1}{M}}^N |\langle a(t) L^\alpha T_{t,\alpha} f, g \rangle| dt \\
&= \lim_{N, M \rightarrow \infty} \left\langle \int_{\frac{1}{M}}^N \tilde{a}(t) L^\alpha T_{t,\alpha} f dt, g \right\rangle \\
&\leq cc_a \|M_{\tilde{a}} f\|_{BMO(L^\alpha)} \|g\|_{H^1} \\
&\leq \frac{cc_a}{(1-\alpha)^2} \|f\|_{L^\infty} \|g\|_{H^1}.
\end{aligned}$$

This shows that $\lim_{N, M \rightarrow \infty} \int_{\frac{1}{M}}^N \langle a(t) L^\alpha T_{t,\alpha} f, g \rangle dt$ exists and $\int_{\frac{1}{M}}^N a(t) L^\alpha T_{t,\alpha} f dt$ weak* converges in $BMO(L^\alpha)$ as $N, M \rightarrow \infty$. So the integration in (64) weak* converges and M_a is well defined for all $f \in L^\infty$ and $a(t)$ satisfying (63). The weak* extension of M_a is then a bounded map from $BMO(L^\alpha)$ to $BMO(L^\alpha)$.

Theorem 2. *Suppose $a(t)$ satisfies (63). M_a extends to a bounded operator from $BMO(L^\alpha)$ to $BMO(L^\alpha)$ for $0 < \alpha < 1$. The estimates are as in Lemma 10.*

Theorem 3. Suppose $T_t = e^{-tL}$ is a weak* continuous semigroup of positive contractions on L^∞ satisfying the $\Gamma^2 \geq 0$ criterion. Then L has a complete bounded $H^\infty(S_\eta)$ calculus on $BMO(\sqrt{L})$ for any $\eta > \frac{\pi}{2}$.

Proof. Given $\alpha \in (\frac{1}{2}, 1)$, let Y^α be the norm closure of $D(L)$ in $BMO(L^\alpha)$. It is easy to check that $T_{t,\alpha} = e^{-tL^\alpha}$ are contractions on Y^α . Then L^α is a densely defined sectorial operator of type $\frac{\pi}{2}$ in Y^α . Lemma 9 and Lemma 10 imply that L^α has a bounded $H^\infty(S_\eta)$ calculus on Y^α for any $\eta > \frac{\pi}{2}$. Note $\Phi(z) = \Psi(z^{\frac{1}{\alpha}}) \in S_\eta$ if $\Psi \in S_{\frac{\eta}{\alpha}}$ and $\Phi(L^\alpha) = \Psi(L)$. We conclude that L has a bounded $H^\infty(S_\eta)$ calculus on Y^α for any $\eta > \frac{\pi}{2\alpha}$. Given $\theta > \frac{\pi}{2}$, choose $\frac{1}{2} < \alpha < 1$ so that $\alpha\theta > \frac{\pi}{2}$. Then L has a bounded $H^\infty(S_\theta)$ calculus on Y^α . Lemma 8 then implies that L has a bounded $H^\infty(S_\theta)$ calculus on $Y^{\frac{1}{2}} \simeq Y^\alpha$ and on $BMO(\sqrt{L})$ for any $\theta > \frac{\pi}{2}$, since $Y^{\frac{1}{2}}$ is weak* dense in $BMO(\sqrt{L})$ and $\Phi(L)$ is the weak* extension of its restriction on $Y^{\frac{1}{2}}$ by definition. The same argument applies to $id \otimes L$. We then obtain the completely bounded $H^\infty(S_\eta)$ calculus as well. \square

3.2 Imaginary Power and Interpolation.

Given $0 < \alpha < 1$, choose $\frac{\pi}{2} < \theta < \frac{\pi}{2\alpha}$. By (61), we have the identities

$$L^\alpha e^{-tL^\alpha} = \frac{1}{2\pi i} \psi(L)^{-1} \int_{\gamma_\theta} z^\alpha \psi(z) e^{-tz^\alpha} (z - L)^{-1} dz, \quad (67)$$

$$L^{i\alpha s} = \frac{1}{2\pi i} \psi(L)^{-1} \int_{\gamma_\theta} z^{i\alpha s} \psi(z) (z - L)^{-1} dz. \quad (68)$$

Note

$$z^{i\alpha s} = \Gamma(1 - is)^{-1} \int_0^\infty t^{-is} z^\alpha e^{-tz^\alpha} dt. \quad (69)$$

Since these integrals converge absolutely, we can exchange the order of the integrations and get

$$L^{i\alpha s} = \Gamma(1 - is)^{-1} \int_0^\infty t^{-is} L^\alpha e^{-tL^\alpha} dt. \quad (70)$$

Lemma 10 implies that

$$\|L^{i\alpha s} f\|_{BMO(L^\alpha)} \leq \frac{c}{(1 - \alpha)^2} \Gamma(1 - is)^{-1} \|f\|_{L^\infty}.$$

Thus for $\alpha > \frac{1}{2}$,

$$\begin{aligned} \|L^{is} f\|_{BMO(L^\alpha)} &\leq c \Gamma\left(1 - i\frac{s}{\alpha}\right)^{-1} \|f\|_{L^\infty} \\ &\leq \frac{c}{(1 - \alpha)^2 (1 + |s|)^{\frac{1}{2}}} \exp\left(\frac{\pi|s|}{2\alpha}\right) \|f\|_{L^\infty}. \end{aligned}$$

Choosing $\alpha = \frac{|s|}{|s|+1}$ for s large, we get

$$\|L^{is}f\|_{BMO(L^\alpha)} \leq c(1+|s|)^{\frac{3}{2}} \exp\left(\frac{\pi|s|}{2}\right) \|f\|_{L^\infty}. \quad (71)$$

The same estimate holds with $\text{bmo}(L^\alpha)$ -norms putting on both sides of (71) because of (65). One can improve such estimates for concrete example of semi-groups, see [?] for example.

Definition 4. We say a weak* continuous semigroup of positive contractions is a *symmetric Markov semigroup* if $\langle T_t f, g \rangle = \langle f, T_t g \rangle$ for $f \in L^\infty, g \in L^1$ and it admits a standard Markov dilation in the sense of [20, page 717].

Remark. The Markov dilation assumption in the above definition holds automatically in many cases. In the commutative case (i.e the underlying von Neumann algebra $\mathcal{M} = L^\infty(M)$), this is due to Rota (see [30, page 106, Theorem 9]). Therefore every weak* continuous semigroup of unital symmetric positive contractions is automatically a symmetric Markov semigroup. In [29] it is proven that this is the case for convolution semigroups on group von Neumann algebras. In [9, 19] it is proven that this holds for the finite von Neumann algebras case. The case of a general semifinite von Neumann algebra is conjectured but there has not been a written proof.

Lemma 11. ([JM12]) Assume that $T_t = e^{-tA}$ (e.g. $A = L^\alpha$) is a symmetric Markov semigroup on a semifinite von Neumann algebra \mathcal{M} . Then, the following interpolation result holds

$$[BMO(A), L_0^1(\mathcal{M})]_{\frac{1}{p}} = L_0^p(\mathcal{M})$$

for $1 < p < \infty$. Here $L_0^p(\mathcal{M}) = L^p(\mathcal{M})/\ker A$.

Since $\|L^{is}\|_{L^2 \rightarrow L^2} = 1$ if L generates a symmetric Markov semigroup, by interpolation, we get from (71) the following result.

Corollary 3. Suppose $T_t = e^{-tL}$ is a symmetric Markov semigroup of operators on a semifinite von Neumann algebra \mathcal{M} and satisfies the $\Gamma^2 \geq 0$ criterion. Then, L has the completely bounded $H^\infty(S_\eta)$ -calculus on L^p for any $\eta > \omega_p = |\frac{\pi}{2} - \frac{\pi}{p}|$, $1 < p < \infty$ and

$$\|L^{is}\|_{L^p \rightarrow L^p} \leq c(1+|s|)^{|\frac{3}{2} - \frac{3}{p}|} \exp\left(\left|\frac{\pi s}{2} - \frac{\pi s}{p}\right|\right), \quad (72)$$

for all $1 < p < \infty$.

Remark. Let us point out that the left-hand side of the inequality on [20, line 4, page 728] misses a “ $\frac{1}{2}$ ”. It should be $\|L^{\frac{is}{2}}f\|$ instead of $\|L^{is}f\|$, because Theorem 3.3 of [20] is for the semigroup generated by \sqrt{L} . So the estimate of the constants $c_{s,p}$ given in [20, Corollary 5.4] is not correct. Also [18] contains a

similar estimate to (72) without assuming the $\Gamma^2 \geq 0$ criterion. Their method is the transference principle and works for L^p only.

Junge, Le Merdy, and Xu ([21]) studied the H^∞ -calculus in the noncommutative setting. In particular, they prove a $H^\infty(S_\eta)$ -calculus property of $L : \lambda_g \mapsto |g|\lambda_g$ on $L^p(\hat{\mathbb{F}}_n)$ for all $1 < p < \infty, \eta > |\frac{\pi}{2} - \frac{\pi}{p}|$.

4 Examples

The “ $\Gamma^2 \geq 0$ ” criterion is known to be satisfied by a large class of semigroups including the heat, Ornstein-Uhlenbeck, and Jacobi semigroups (see [2]). The results proved in this article apply to all of them. The main example in the noncommutative setting, is the semigroup of operators on a group von Neumann algebra, generated from a conditionally negative function on the underlying group (see Example 4). We will analyze a few of them in the following.

Example 1. Let $-L = \Delta$ be the Laplace-Beltrami operator on a complete Riemannian manifold with nonnegative Ricci curvatures. Then the associated heat semigroup $T_t = e^{-tL}$ is symmetric Markovian and satisfies the $\Gamma^2 \geq 0$ criterion. All the theorems of this article hold for L , and it has bounded $H^\infty(S_\eta)$ calculus on $BMO(\sqrt{L})$ for any $\eta > \frac{\pi}{2}$.

In the special case that $L = -\partial_x^2$ the Laplacian on Euclidean space \mathbb{R}^n , the $BMO(L)$, $bmo(L)$, and $BMO(\sqrt{L})$ spaces are all equivalent to the classical BMO space of all functions $f \in L^1(\mathbb{R}^n, \frac{1}{1+|x|^2}dx)$ with a finite BMO norm,

$$\|f\|_{BMO(\mathbb{R}^n)} = \sup_{B \subset \mathbb{R}^n} (E_B |f - E_B f|^2)^{\frac{1}{2}} < \infty.$$

Here the supremum runs on all balls (or cubes) in \mathbb{R}^n and $E_B = \frac{1}{|B|} \int_B f dx$ denotes the mean value operator. This can be verified by the integral representation of $T_t, T_{t, \frac{1}{2}}$, the convexity of $|\cdot|^2$ and the fact that $|E_B f - E_{kB} f| \lesssim \log k \|f\|_{BMO(\mathbb{R}^n)}$. By Lemma 8 we then get the equivalence between $BMO(\mathbb{R}^n)$ and $BMO(L^\alpha)$ for all $0 < \alpha \leq 1$.

Example 2. Let $L = \partial_x$ on \mathbb{R} . Then $T_t = e^{-tL}$ is the translation operator sending $f(\cdot)$ to $f(\cdot - t)$. It is a Markov semigroup and the $\Gamma^2 \geq 0$ criterion holds trivially. The $BMO(L)$ space is equivalent to L_0^∞ and the $bmo(L)$ (semi)norm vanishes. For any $0 < \alpha < 1$, $BMO(L^\alpha)$ is equivalent to the classical $BMO(\mathbb{R}^n)$ space. Indeed, by the subordination formula, we get the following integral representation for $T_{t, \frac{1}{2}} = e^{-t\sqrt{L}}$:

$$T_{t, \frac{1}{2}} f(x) = \frac{1}{2\sqrt{\pi}} \int_0^\infty f(x-s) t e^{-\frac{t^2}{4s}} s^{-\frac{3}{2}} ds.$$

From this, it is easy to check that, for $I_{x,k} = [x - 2^k \frac{t^2}{4}, x - 2^{-k} \frac{t^2}{4}]$, $k \in \mathbb{N}$,

$$c^{-1} E_{I_{x,1}} |f| \leq T_{t, \frac{1}{2}} |f|(x) \leq c \sum_k 2^{-\frac{k}{2}} E_{I_{x,k}} |f|.$$

After an elementary calculation and using the fact that $|E_B f - E_{k_B} f| \lesssim \log k \|f\|_{BMO(\mathbb{R}^n)}$, one can see that $\|\cdot\|_{BMO(\sqrt{L})} \simeq \|\cdot\|_{BMO(\mathbb{R}^n)}$, thus $\|\cdot\|_{BMO(L^\alpha)} \simeq \|\cdot\|_{BMO(\mathbb{R}^n)}$ for all $0 < \alpha < 1$ by Lemma 8.

By Theorem 3, L has $H^\infty(S_\eta)$ -calculus on $BMO(\sqrt{L}) \simeq BMO(\mathbb{R}^n)$ for any $\eta > \frac{\pi}{2}$. It is easy to see that

$$L^{is} = P_+ e^{\frac{-s\pi}{2}} \Delta^{\frac{is}{2}} + P_- e^{\frac{s\pi}{2}} \Delta^{\frac{is}{2}}.$$

So L does not have $H^\infty(S_\theta)$ -calculus on $BMO(L) \simeq L^\infty(\mathbb{R})/\mathbb{C}$ for any positive θ and

$$\|L^{is}\|_{BMO \rightarrow BMO} \simeq e^{\frac{\pi|s|}{2}} \|\Delta^{\frac{is}{2}}\|_{BMO \rightarrow BMO}$$

for $|s|$ large. This also shows that L cannot have bounded $H^\infty(S_\eta)$ -calculus on $BMO(\mathbb{R})$ for any $\eta \leq \frac{\pi}{2}$.

Example 3. Let $-L = \frac{\partial^2}{x} - x \cdot \partial_x$ be the Ornstein-Uhlenbeck operator on $(\mathbb{R}^n, e^{-|x|^2} dx)$. Let $O_t f = O_{t,1} = e^{-tL}$. O_t is a symmetric Markov semigroup with respect to the Gaussian measure $d\mu = e^{-|x|^2} dx$ and satisfies the $\Gamma^2 \geq 0$ criterion. Theorem 3 says that $L = -\frac{\partial^2}{x} + x \cdot \partial_x$ has bounded $H^\infty(S_\eta)$ -calculus on $BMO(\sqrt{L})$ for any $\eta > \frac{\pi}{2}$.

Mauceri and Meda (see [27]) introduced the following BMO space for the Ornstein-Uhlenbeck semigroup

$$\|f\|_{BMO(MM)} = \sup_{r_B \leq \min\{1, \frac{1}{|c_B|}\}} (E_B^\mu |f - E_B^\mu f|^2)^{\frac{1}{2}}, \quad (73)$$

with r_B, c_B the radius and the center of B , and $E_B^\mu = \frac{1}{\mu(B)} \int \cdot d\mu$ the mean value operator with respect to the Gaussian measure $d\mu$. Note, for the balls B satisfying $r_B \leq \min\{1, \frac{1}{|c_B|}\}$, we have the equivalence $E_B^\mu |f| \simeq E_B |f|$. One may replace E_B^μ by E_B , the mean value operator with respect to the Lebesgue measure dx in (73). The resulted BMO norms are equivalent to each other. From the integral presentation

$$O_t(f) = \frac{1}{(\pi - \pi e^{-2t})^{\frac{n}{2}}} \int_{\mathbb{R}^n} \exp\left(-\frac{|e^{-t}x - y|^2}{1 - e^{-2t}}\right) f(y) dy, \quad (74)$$

one easily see that, for $t \leq 4$ and $\sqrt{t}|x| \leq 1$,

$$\begin{aligned} O_t |f|(x) &\geq \frac{1}{(\pi - \pi e^{-2t})^{\frac{n}{2}}} \int_{B(x, \sqrt{t})} \exp\left(-\frac{2|x - y|^2}{1 - e^{-2t}}\right) f(y) dy \\ &\geq c_n E_{B(x, \sqrt{t})} |f|(x). \end{aligned} \quad (75)$$

Note $E_{B(x, \sqrt{t})} |f| \leq c_n E_{B(x, \sqrt{s})} |f|$ for all $t < s < 2t$. We then have from (75) that, for $O_{t, \frac{1}{2}} = e^{-tL^{\frac{1}{2}}}$, $t \leq 1, tx \leq 1$,

$$O_{t, \frac{1}{2}} |f|(x) = \int_0^\infty O_s |f|(x) \phi_{t, \frac{1}{2}}(s) ds \geq \frac{c}{\sqrt{t}} \int_{t^2}^{4t^2} O_s |f|(x) ds \geq c_n E_{B(x, t)} |f|(x).$$

We then easily get

$$4O_{t,\alpha}|f - O_{t,\alpha}f|^2(x) \geq c_n E_{B(x,t\frac{1}{2\alpha})}|f - E_{B(x,t\frac{1}{2\alpha})}f(x)|^2(x), \quad (76)$$

by the convexity of $|\cdot|^2$, for $\alpha = \frac{1}{2}, 1$. Therefore,

$$\|\cdot\|_{BMO(MM)} \lesssim \|\cdot\|_{BMO(L)}, \|\cdot\|_{bmo(L)}, \|\cdot\|_{BMO(\sqrt{L})},$$

and by Lemma 8,

$$\|\cdot\|_{BMO(MM)} \lesssim \|\cdot\|_{BMO(L^\alpha)}$$

for all $0 < \alpha \leq 1$. By Theorem 3, the Ornstein-Uhlenbeck operator $L = -\frac{\partial_x^2}{2} + x \cdot \partial_x$ has bounded $H^\infty(S_\eta)$ calculus from $L^\infty(\mathbb{R}^n)$ to Mauceri-Meda's $BMO(MM)$ for any $\eta > \frac{\pi}{2}$.

Let $f(y) = \frac{1}{\sqrt{4\pi s}} \exp(-\frac{|y|^2}{4s})$, with $s > 100$. We have

$$\begin{aligned} & (O_t|f|^2 - |O_t f|^2)(x) \\ &= \frac{1}{4\pi\sqrt{(s+2v)s}} \exp\left(-\frac{|e^{-t}x|^2}{2s+4v}\right) - \frac{1}{4\pi(s+v)} \exp\left(-\frac{|e^{-t}x|^2}{2s+2v}\right) \\ &= \left(\frac{1}{4\pi\sqrt{(s+2v)s}} - \frac{1}{4\pi(s+v)}\right) \exp\left(-\frac{|e^{-t}x|^2}{2s+4v}\right) \\ & \quad + \frac{1}{4\pi(s+v)} \left(\exp\left(-\frac{|e^{-t}x|^2}{2s+4v}\right) - \exp\left(-\frac{|e^{-t}x|^2}{2s+2v}\right)\right) \\ &\lesssim \frac{1}{s^3} + \frac{1}{s^2} \lesssim \frac{1}{s^2}. \end{aligned}$$

On the other hand, for $v = \frac{1-e^{-2t}}{4}$, $v' = \frac{1-e^{-4t}}{4}$,

$$(O_t f - O_{2t} f)(x) = \frac{1}{\sqrt{4\pi(s+v)}} e^{-\frac{|e^{-t}x|^2}{4s+4v}} - \frac{1}{\sqrt{4\pi(s+v')}} e^{-\frac{|e^{-2t}x|^2}{4s+4v'}}$$

For $x^2 = e^{2t}(4s+4v)$, $t = 10$, we get

$$\begin{aligned} |(O_t f - O_{2t} f)(x)| &\geq \left| \frac{1}{\sqrt{4\pi(s+v)}} e^{-1} - \frac{1}{\sqrt{4\pi(s+v')}} e^{-\frac{1}{100}} \right| \\ &\geq \frac{1}{2\sqrt{4\pi(s+v')}} \geq \frac{1}{10\sqrt{s}}. \end{aligned}$$

So,

$$\|f\|_{BMO(L)} \geq \sup_{t>0} \|O_t f - O_{2t} f\|_{L^\infty} \geq \frac{\sqrt{s}}{5} \|f\|_{bmo(L)}.$$

Therefore, the $BMO(L)$ and $bmo(L)$ -norms are not equivalent for the Ornstein-Uhlenbeck semigroup, by letting $s \rightarrow \infty$. This shows that one can not extend Lemma 8 to the case of $\alpha = 1$.

Example 4. Let (G, μ) be a locally compact unimodular group with its Haar measure. Let $\lambda_g, g \in G$ be the translation-operator on $L^2(G)$ defined as

$$\lambda_g(f)(h) = f(g^{-1}h).$$

The so-called group von Neumann algebra $L^\infty(\hat{G})$ is the weak* closure in $B(L_2(G))$ of the operators $f = \int_G \hat{f}(g)\lambda_g d\mu(g)$ with $\hat{f} \in C_c(G)$. The canonical trace τ on $L^\infty(\hat{G})$ is defined as $\tau f = \hat{f}(e)$. If G is abelian, then $L^\infty(\hat{G})$ is the canonical L^∞ space of functions on the dual group \hat{G} . In particular, if $G = \mathbb{Z}$, the integer group, then $\lambda_k = e^{ikt}, k \in \mathbb{Z}$ and $L^p(\hat{\mathbb{Z}}) = L^p(\mathbb{T})$, the function space on the unit circle. Please refer to [28] for details on noncommutative L^p spaces.

Let φ be a scalar valued function on G . We say φ is *conditionally negative* if $\varphi(g^{-1}) = \varphi(g)^*$ and

$$\sum_{g,h} \overline{a_g} a_h \varphi(g^{-1}h) \leq 0 \quad (77)$$

for any finite collection of coefficients $a_g \in \mathbb{C}$ with $\sum_g a_g = 0$. Schöenberg's theorem says that

$$T_t : \lambda_g = e^{-t\varphi(g)} \lambda_g$$

extends to a Markov semigroups of operators on the group von Neumann algebra $L^\infty(\hat{G})$ if and only if φ is a conditionally negative function with $\varphi(e) = 0$. The negative generator of the semigroup is the unbounded map $L : \lambda_g \mapsto \varphi(g)\lambda_g$ which is weak* densely defined on $L^\infty(\hat{G})$.

Let $K_\varphi(g, h) = \frac{1}{2}(\varphi(g) + \varphi(h) - \varphi(g^{-1}h))$, the Gromov form associated with φ . Then one can directly verify from (77) that K_φ is a positive definite function on $G \times G$. Thus K_φ^2 is a positive definite function too. This is equivalent to the $\Gamma^2 \geq 0$ criterion for T_t , and therefore Theorem 3 applies to all such $(T_t)_t$'s. If in addition, φ is real valued, then (T_t) is a symmetric Markov semigroup. We then obtain the following corollary.

Corollary 4. *Let G be a locally compact unimodular group. Suppose φ is a conditionally negative function on G with $\varphi(e) = 0$. Let L be the weak* densely defined linear map on $L^\infty(\hat{G})$ such that $L(\lambda_g) = \varphi(g)\lambda_g$. Then,*

(i) *For any $\eta > \frac{\pi}{2}$ and any bounded analytic Φ on S_η , the map $\Phi(L) : \lambda_g \mapsto \Phi(\varphi(g))\lambda_g$ extends to a completely bounded operator on $BMO(\sqrt{L})$ and $\|\Phi(L)\| \leq C_\eta \|\Phi\|_\infty$.*

(ii) *Suppose in addition that φ is real valued. If Φ is a bounded analytic function on S_η with $\eta > |\frac{\pi}{2} - \frac{\pi}{p}|$, then the map $\Phi(L)$ extends to a completely bounded operator on $L^p(\hat{G})$ for $1 < p < \infty$.*

Remark. Corollary 4 (i) was proved in [25] for $L : \lambda_g \mapsto \sqrt{\varphi(g)}\lambda_g$ with φ a symmetric conditionally negative function on G .

Example 5. Let $G = \mathbb{F}_\infty$ be the nonabelian free group with a countably infinite number of generators. Let $|g|$ be the reduced word length of $g \in G$.

Then $\varphi : g \rightarrow |g|$ is a conditionally negative function (see [15]) and $L : \lambda_g \mapsto |g|\lambda_g$ generates a symmetric Markov semigroup on the free group von Neumann algebra. Fix $\theta \in (\frac{\pi}{2}, \pi)$, let $\Phi(z) = (\ln(z+2))^{-1}$ for $z \in S_\theta$. Then $\Phi \in H^\infty(S_\theta)$. Corollary 4 then implies that the Fourier multiplier

$$\lambda_g \mapsto \frac{1}{\ln(|g|+2)} \lambda_g$$

extends to a bounded operator on $\text{BMO}(\sqrt{L})$. By the interpolation result Lemma 11, we conclude that this multiplier is bounded on $L^p(\hat{\mathbb{F}}_\infty)$ with constant $\lesssim \frac{p^2}{p-1}$.

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